PROBABILISTIC GENE REGULATORY NETWORKS, ISOMORPHISMS OF MARKOV CHAINS

MARÍA ALICIA AVIÑÓ

Abstract. In this paper we study homomorphisms of Probabilistic Regulatory Gene Networks(PRN) introduced in [2]. The model PRN is a natural generalization of the Probabilistic Boolean Networks (PBN), introduced by I. Shmulevich, E. Dougherty, and W. Zhang in [14], that has been using to describe genetic networks and has therapeutic applications, see [15]. In this paper, our main objectives are to apply the concept of homomorphism and ϵ-homomorphism of probabilistic regulatory networks to the dynamic of the networks. The meaning of ϵ is that these homomorphic networks have similar distributions and the distance between the distributions is upper bounded by ϵ. Additionally, we prove that the class of PRN together with the homomorphisms form a category with products and coproducts. Projections are special homomorphisms, and they always induce invariant subnetworks that contain all the cycles and steady states in the network. Here, it is proved that the ϵ-homomorphism for $0 < \epsilon < 1$ produce simultaneous Markov Chains in both networks, that permit to introduce the concept of ϵ-isomorphism of Markov Chains, and similar networks.

Introduction

Genes can be understanding in their complexity behavior using models according with their discrete or continuous action. Developing computational tools permits describe gene functions and understand the mechanism of regulation [8, 9]. *This understanding will have a significant impact on the development of techniques for drugs testing and therapeutic intervention for treating human diseases*[7, 13, 15].

We focus our attention in the discrete structure of genetic regulatory networks, instead of, its dual moving continuo-discrete. Probabilistic Gene Regulatory Network(PRgN) is a natural generalizations of the model Probabilistic Boolean Network (PBN), introduced by I. Shmulevich, E. Dougherty, and W. Zhang in [14]. The mathematical background of the model PgRN, is introduced here, for simplicity we work with functions defined over a set X to itself, with probabilities assigned to these functions. X is a set of states of genes, for example $X = \{0,1\}^n$, if our network is a Boolean network. Working in this way, we can observe the dynamic of the network indeed focus our attention in the description of functions. The set

X can be a subset of $\{0,1\}^n$, and we can extend some classical ideas to regulatory network, such as invariant subnetworks, automorphisms group, etc. In particular if X is a vector space over a finite field, the functions are lineal functions, then we can use linear algebra to describe the state space. Mapping are important in the study of networks, because they permit to recognize subnetworks, in particular determine when two networks are similar or equivalent. Special mappings are homomorphisms and ϵ-homomorphisms, we use both to describe subnetworks and similar networks. An homomorphism transform a network to another in such a way the discrete structure giving by the first network can lives in part of the other one, or these two networks are very similar but no equals, in particular in the probabilistic way. An ϵ-homomorphism is the same but with the condition that the probability distributions of the networks are close, and we use a preestablishes $0 < \epsilon < 1$ as a distance between the probabilities. For concept of homomorphism of discrete dynamical systems see [3, 4, 12].

1. PRELIMINARIES CONCEPTS:FINITE DYNAMICAL SYSTEMS, PROBABILISTIC BOOLEAN NETWORKS AND PROBABILISTIC REGULATORY NETWORKS

Two finite dynamical systems (X, f) and (Y, g) are isomorphicas(or equivalents) if there exists a bijection $\phi : X \to Y$ such that $\phi \circ f = g \circ \phi$, (or $f = \phi^{-1} \circ g \circ \phi$). If ϕ is not a bijection map then ϕ is an homomorphism.

If $Y \subset X$ is such that $f(Y) \subset Y$ then $(Y, f|_y)$ is a sub-FDS of (X, f), where $f|_Y$ is the map restricted to Y. There exists naturally an injective morphism from Y to X called inclusion and denoted by ι. The state space of a FDS (X, f), is a digraph whit vertices the set X, and with an arrow from u to v if $f(u) = v$.

For example, the FDSs $X = (\{0,1\}^2, f_1(x, y) = (xy, y))$, and $Y = (\{0,1\}^2, f_2(x, y) = (x, (x+1)y))$ are isomorphics, because their state spaces are isomorphics.

$$
\begin{array}{cccc}
(1,0) & \to^{f_1} & (0,0) & (0,1) \\
& & \circlearrowright & \circlearrowright^{f_1} \quad \curvearrowright^{f_1} \\
& & & (1,1)
\end{array}
$$

$$
\begin{array}{cccc}
(1,1) & \to^{f_2} & (1,0) & (0,0) \\
& & \circlearrowright & \circlearrowright^{f_2} \quad \curvearrowright^{f_2} \\
& & & (0,1)
\end{array}
$$

In fact, the isomorphism $\phi : \{0,1\}^2 \to \{0,1\}^2$ is the bijection $\phi(1,0) = (1,1)$, $\phi(0,0) = (1,0)$, $\phi(0,1) = (0,0)$, and $\phi(1,1) = (0,1)$. The following is an example of homomorphism (inclusion) with $Z = \{\{(0,0),(1,0)\}, f_1\}$.

$$
\begin{array}{ccccccc}
(1,0) & & & (1,0) & & (0,1) & \\
\downarrow f_1 & \hookrightarrow^{\iota} & & \downarrow f_1 & & \circlearrowright^{f_1} \quad \curvearrowright^{f_1} \\
(0,0) & \circlearrowright & & (0,0) & \circlearrowright & (1,1)
\end{array}
$$

A Probabilistic Boolean Network $\mathcal{A} = (V, F, C)$ is defined by the following sort (type) of objects [14, ?]: a set of nodes (genes) $V = \{x_1, \ldots, x_n\}$, $x_i \in \{0,1\}$, for all i; a family $F = \{F_1, F_2, \ldots, F_n\}$ of ordered sets $F_i = \{f_1^{(i)}, f_2^{(i)}, \ldots, f_{\ell(i)}^{(i)}\}$ of Boolean functions $f_j^{(i)} : \{0,1\}^n \to \{0,1\}$, for all j called predictors; and a list $C = (C_1, \ldots, C_n)$, $C_i = \{c_1^{(i)}, \ldots, c_{\ell(i)}^{(i)}\}$, of selection probabilities. The selection probability that the function $f_j^{(i)}$ is used for the vertex i is $c_j^{(i)} = Pr\{f^{(i)} = f_j^{(i)}\}$. The dynamic of the PBN is given by a vector of functions $\mathbf{f}_k = (f_{k_1}^{(1)}, f_{k_2}^{(2)}, \ldots, f_{k_n}^{(n)})$

for $1 \le k_i \le l(i)$, and $f_{k_i}^{(i)} \in F_i$, where $k = [k_1, \ldots, k_n]$, $1 \le k_i \le l(i)$. The map $\mathbf{f}_k : \{0,1\}^n \to \{0,1\}^n$ acts as a transition function. Each variable $x_i \in \{0,1\}^n$ represents the state of the vertex i. All functions are updated synchronously. At every time step, one of the functions is selected randomly from the set F_i according to a predefined probability distribution. The selection probability that the transition function $\mathbf{f}_k = (f_{k_1}^{(1)}, f_{k_2}^{(2)}, \ldots, f_{k_n}^{(n)})$ is used to go from the state $u \in \{0,1\}$ to another state $\mathbf{f}_k(u) = v \in \{0,1\}^n$ is given by

$$c_{\mathbf{f}_k} = \prod_{i=1}^{n} c_{k_i}^{(i)}.$$

The dynamical transition structure of a PBN can be described by a Markov chain with fixed transition probabilities. There are two digraphs structures associated with a PBN: the low-level digraph Γ, consisting of genes functions essentiality relations; and the high-level digraph which consists of the states of the system and the transitions between states. The matrix T associated to the high level digraph formed by placing $p(u,v)$ in row u and column v, where u, $v \in \{0,1\}^n$ is called the transition probability matrix or chain matrix, $p(u,v) = \sum_{\mathbf{f}_k | \mathbf{f}_k(u) = v} c_{\mathbf{f}_k}$.

1.1. Probabilistic Regulatory Gene Networks.

A Probabilistic Gene Regulatory Network (PRN) is a triple $\mathcal{X} = (X, F, C)$ where X is a finite set and $F = \{f_1, \ldots, f_n\}$ is a set of functions from X into itself, with a list $C = (c_1, \ldots, c_n)$ of selection probabilities, where $c_i = p(f_i)$, [2, 1] We associate with each PRN a weighted digraph, whose vertices are the elements of X, and if $u, v \in X$, there is an arrow going from u to v for each function f_i such that $f_i(u) = v$, and the probability c_i is assigned to this arrow. This weighted digraph will be called the state space of $\mathcal{X}$. In this paper, we use the notation PRN for one or more networks.

Example 1.1. *If* $X = \{0,1\}^2$, $F = \{f_1(x,y) = (x,y), f_2(x,y) = (x,0),$ $f_3(x,y) = (1,y), f_4(x,y) = (1,0)\}$; and $C = \{.46, .21, .22, .11\}$, the state space of $\overline{\mathcal{X}} = (X, F, C)$ is the following:

$$\overset{.67}{\circlearrowright} (0,0) \xleftarrow{.21} (0,1) \circlearrowright^{.46} \qquad T = \begin{bmatrix} .67 & 0 & .33 & 0 \\ .21 & .46 & .11 & .22 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & .32 & .68 \end{bmatrix}$$
$$\overset{.33}{\downarrow} \quad \swarrow_{.11} \downarrow^{.22}$$
$$\overset{1}{\circlearrowright} (1,0) \xleftarrow{.32} (1,1) \circlearrowright^{.68}$$

1.2. Homomorphisms and ϵ-homomorphisms of PRN.

If C is a set of selection probabilities we denote by χ the characteristic function over C. That is $\chi : C \cup \{0\} \to \{0,1\}$ such that $\chi(c) = 1$, if $c \ne 0$ and $\chi(0) = 0$. Let $\mathcal{X}_1 = (X_1, F = (f_i)_{i=1}^n, C)$ and $\mathcal{X}_2 = (X_2, G = (g_j)_{j=1}^m, D)$ be two PRN.

Definition 1.2 (Homomorphisms of PRN). *A map* $\phi : X_1 \to X_2$ *is an* **homomorphism** *from* $\mathcal{X}_1$ *to* $\mathcal{X}_2$, *if for all* f_i *there exists a* g_j, *such that for all* u, v *in* $\mathcal{X}_1$,

$$(1)\ \phi \circ f_i = g_j \circ \phi;\ \text{ and }\ (2)\ \chi(d_{g_j}(\phi(u), \phi(v))) \ge \chi(c_{f_i}(u,v)).$$

$$\begin{array}{ccc} X_1 & \xrightarrow{f_i} & X_1 \\ \phi \downarrow & & \downarrow \phi \\ X_2 & \xrightarrow{g_j} & X_2 \end{array}$$

(3) Condition for ϵ-Homomorphism: The distributions of probabilities following the homomorphism are enough close. An ϵ- homomorphism is an homomorphism that satisfies the condition, for all i, j, $max|p(u_i, u_j) - p(\phi(u_i), \phi(u_j))| \leq \epsilon$, where $\epsilon > 0$ is a real number that we previously determine for the applications.

If $\phi : X_1 \to X_2$ is a bijective map, and for all f_i, g_j, u, and v in $\mathcal{X}_1$; $d_{g_j}(\phi(u), \phi(v)) = c_{f_i}(u, v)$, then ϕ is an isomorphism.

Example 1.3.

If $\overline{\mathcal{X}} = (X; F; C)$ is the PRN in Example 1, and $\mathcal{X}_1 = (X; F' = \{f_1, f_2, f_3\}; C' = \{.47, .28, .25\})$ is a new PRN over the same set X with different probabilities and only three functions.

$$\mathcal{X}_1 \qquad\qquad\qquad \overline{\mathcal{X}}$$

$$^{.75} \circlearrowleft (0,0) \xleftarrow{.28} (0,1) \overset{.47}{\circlearrowright} \quad \overset{\phi}{\hookrightarrow} \quad ^{.67} \circlearrowleft (0,0) \xleftarrow{.21} (0,1) \overset{.46}{\circlearrowright}$$

$$^{.25} \downarrow \qquad\qquad \downarrow^{.25} \qquad\qquad\qquad ^{.33} \downarrow \quad \swarrow_{.11} \downarrow^{.22}$$

$$\circlearrowleft^1 (1,0) \xleftarrow{.28} (1,1) \circlearrowleft^{.72} \qquad \circlearrowleft^1 (1,0) \xleftarrow{.32} (1,1) \circlearrowleft^{.68}$$

$$T_1 = \begin{bmatrix} .75 & 0 & .25 & 0 \\ .28 & .47 & 0 & .25 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & .28 & .72 \end{bmatrix}, \overline{T} = \begin{bmatrix} .67 & 0 & .33 & 0 \\ .21 & .46 & .11 & .22 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & .32 & .68 \end{bmatrix}$$

The homomorphism $\phi : \mathcal{X}_1 \to \overline{\mathcal{X}}$ is a bijective map, $\phi(x) = x$, over the set of states, but an inclusion over the set of arrows, because the arrow going from $(0,1)$ to $(1,1)$ in $\overline{\mathcal{X}}$ doesn't appear in $\mathcal{X}_1$. The first condition for homomorphism is obvious. The condition (2) holds, because the inclusion of arrows. The two transition matrices are connected by this inclusion, since if the place ij in the first matrix $\neq 0$ then this place is $\neq 0$ in the second network too. The two PRN are not isomorphics because the probabilities are not equals. In order to determine ϵ for the homomorphism, we use the transition matrices. In this example $\epsilon = .11$.

$$T_1 - \overline{T} = \begin{bmatrix} .08 & 0 & -.08 & 0 \\ .07 & .01 & -.11 & .03 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -.04 & .04 \end{bmatrix}$$

If the homomorphism is a bijective map like here, the transition matrices T_1 and T_2 have the same order, and $\sum_{i=1}^{n}(T_1 - T_2)_{ij} = 0$, for $j = 1, n$

2. Applications to Markov Chains, ϵ-Similar Networks

Two PRN are ϵ-similar if there exists a bijective homomorphism ϕ between them, such that ϕ^{-1} is also an homomorphism. Observe that ϕ and ϕ^{-1} have the same ϵ.

Example 2.1.

The following networks are .005-homomorphics with the injective homeomorphism $\phi : \{0,1\}^2 \to \{0,1\}^3$, giving by $\phi(x,y) = (x,y,1)$. We can see in this example, that the network $\mathcal{X}_1$ is an ϵ-subnetwork of $\mathcal{X}_2$, the special subnetwork

$\phi(\mathcal{X}_1)$ is an invariant subnetwork of $\mathcal{X}_2$. The networks, and the transition matrices are the following:

$$
\mathcal{X}_1 \qquad
00 \; \underset{.549}{\overset{.111,.451}{\longleftrightarrow}} \; 10 \; \overset{.444}{\circlearrowleft}
$$

$$
T_1 = \begin{bmatrix} 0 & .549 & .451 & 0 \\ 0 & .338 & 0 & .662 \\ .111 & .445 & .444 & 0 \\ 0 & .013 & 0 & .987 \end{bmatrix}
$$

(with $10 \overset{.445}{\downarrow} 01 \overset{.338}{\circlearrowleft}$, $01 \overset{.013.662}{\underset{\uparrow\downarrow}{}} 11 \overset{.987}{\circlearrowleft}$)

$$
\mathcal{X}_2
$$

$$
000 \; \underset{.451}{\overset{.549}{\longrightarrow}} \; 100 \; \overset{.005}{\rightarrow} \; 001 \; \underset{.544}{\overset{.456,.113}{\longleftrightarrow}} \; 101 \; \overset{.448}{\circlearrowleft}
$$

$$
100 \overset{.995}{\downarrow} \; 010 \overset{.622}{\rightarrow} 110 \; ; \quad 101 \overset{.439}{\downarrow} \; 011 \overset{.337}{\circlearrowleft}
$$

$$
010 \overset{.378}{\circlearrowleft} \; ; \quad 110 \; \circlearrowleft^{.998} \; ; \quad 011 \overset{.663.011}{\underset{\downarrow\uparrow}{}} 111 \overset{.989}{\circlearrowleft}
$$
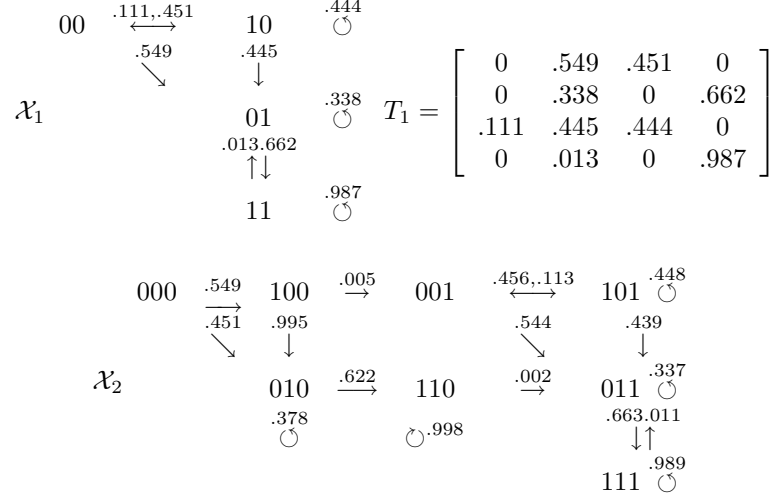
Ordering the elements in the following way

$$
\{(0,0,0), (0,1,0), (1,0,0), (1,1,0), (0,0,1), (0,1,1), (1,0,1), (1,1,1)\}
$$

$$
T_2 = \begin{bmatrix}
0 & .451 & .549 & 0 & 0 & 0 & 0 & 0 \\
0 & .378 & 0 & .622 & 0 & 0 & 0 & 0 \\
0 & .995 & 0 & 0 & .005 & 0 & 0 & 0 \\
0 & 0 & 0 & .998 & 0 & .002 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & .544 & .456 & 0 \\
0 & 0 & 0 & 0 & 0 & .337 & 0 & .663 \\
0 & 0 & 0 & 0 & .113 & .448 & .439 & 0 \\
0 & 0 & 0 & 0 & 0 & .011 & 0 & .989
\end{bmatrix}.
$$

Where, $T_\phi = \begin{bmatrix} 0 & .544 & .456 & 0 \\ 0 & .337 & 0 & .663 \\ .113 & .448 & .439 & 0 \\ 0 & .011 & 0 & .989 \end{bmatrix}$. Observe that,

$$
T_1 - T_\phi = \begin{bmatrix}
0 & .005 & -.005 & 0 \\
0 & .001 & 0 & -.001 \\
-.002 & -.003 & .005 & 0 \\
0 & .002 & 0 & -.002
\end{bmatrix}
$$

As a consequence, we obtain $max|(T_1)_{ij} - (T_\phi)_{ij}| \le .005$, then the networks are .006-homeomorphiscs. The steady state of T_1 is $\pi_1 = (0, .01926, 0, .98074)$, and the steady state of T_ϕ is $\pi_\phi = (0, .01632, 0, .98368)$. We can see that $|\pi_1 - \pi_\phi| = max_i|\pi_1(i) - \pi_\phi(i)| < .004$. Meanwhile for T_2 we have, $\pi_2 = (0, 0, 0, .01632, 0, 0, 0, .98368)$. We can observe that with the new order in the elements of $\mathbb{Z}_2{}^3$, the transition matrix T_ϕ is an invariant submatrix of T_2, that is $T_2 = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_\phi \end{bmatrix}$. In the above example, $\mathcal{X}_1$ and $\phi(\mathcal{X}_1)$ are ϵ-similar.

To introduce the following theorem we use the transition matrices in this example, and we can observe the following calculation

$$T_1^2 - T_\phi{}^2 = \begin{bmatrix} -.001467 & -.00136 & .00006 & .00277 \\ 0 & .00199 & 0 & -.00199 \\ -.00232 & -.00019 & .00295 & -.00243 \\ 0 & .002639 & 0 & -.00263 \end{bmatrix},$$

therefore $max|(T_1^2)_{ij} - (T_\phi^2)_{ij}| \le .003$.

$$T_1^3 - T_\phi{}^3 = \begin{bmatrix} -.000394 & -.00044 & .00011 & .00073 \\ 0 & .002525 & 0 & -.00253 \\ -.000161 & .00156 & .00213 & -.00353 \\ 0 & .002843 & 0 & -.002843 \end{bmatrix},$$

and $max|(T_1^3)_{ij} - (T_\phi^3)_{ij}| \le .004$.

Theorem 2.2. *If $\phi : \mathcal{X}_1 \to \mathcal{X}_2$ is an ϵ-homomorphism, then the transition matrices T_1 and T_ϕ satisfy the condition*

$$max|(T_1{}^n)_{ij} - (T_\phi{}^n)_{ij}| \le \epsilon$$

for all possible i and j, and all $n > 1$. If the homomorphism is injective and $\epsilon < 1$, the steady state of T_1 and the steady state of T_ϕ are close, that is satisfy $|\pi_1 - \pi_\phi| = max_i|\pi_1(i) - \pi_\phi(i)| \le \epsilon$.

Proof. Since $(T_1{}^n)_{ij} = p(u_i, f^n(u_i))$ when $f^n(u_i) = u_j$, therefore we have the following:

$$|p(u, f^2(u)) - p(\phi(u), \phi(f^2(u)))| =$$

$$|p(u, f(u))p(f(u), f^2(u)) - p(\phi(u), \phi(f(u)))p(\phi(f(u)), \phi(f^2(u)))| \le$$

$$|p(f(u), f^2(u))||p(u, f(u)) - p(\phi(u), \phi(f(u)))| +$$

$$|p(\phi(u), \phi(f(u)))||p(f(u), f^2(u)) - p(\phi(f(u)), \phi(f^2(u)))| \le \epsilon$$

Using this property, mathematical induction over n, and the cases giving for the possible values of the probabilities $p(u, v)$, we can conclude that our aim holds. $\square$

An ϵ-homomorphism between two PRN determines a correspondence between the Markov Chains of these two networks. The following new definition is connected in some way with the idea of bisimulation for two TDMC (Time Discrete Markov Chains) studied and introduced in [6]. Two TDMC are equivalent by a bisimulation if they simulate the same problem with two different set of probabilities. Here, we introduce the concept of two similar TDMC

Definition 2.3. *The two TDMC of the same size $n \times n$: $\{T_1, T_1^2, T_1^3, \ldots\}$, and $\{T_2, T_2^2, T_2^3, \ldots\}$ are ϵ-similar or ϵ-isomorphic if*

(1) *there exists an $\epsilon \in \mathbb{R}$ enough small, such that $T_1^m - T_2^m = (t_{ij})_{n \times n}$ satisfies that $|t_{ij}| < \epsilon$, and $\sum_{i=1}^n t_{ij} = 0$, for all m,*

(2) *$\chi(T_1^m)_{ij} = \chi(T_2^m)_{ij}$, for all m, where χ is the characteristic function.*

That is, simulating the networks by these two TDMC are ϵ-similar.

In the above example, the TDMC generated by T and T_ϕ are .005-similar, and the networks simulated by them are .005-similar.

3. Construction of Probabilistic Regulatory Networks, examples

Sum of two PRN

Let $\mathcal{X}_1 = (X_1, F = (f_i)_{i=1}^n, C)$ and $\mathcal{X}_2 = (X_2, G = (g_j)_{j=1}^m, D)$ be two PRN. The sum $\mathcal{X}_1 \oplus \mathcal{X}_2 = (X_1 \dot\cup X_2, F \vee G, C \vee D)$ is a PRN where

(1) $X_1 \dot\cup X_2$ is the disjoint union of X_1 and X_2.
(2) the function $h_{ij} = (f_i \vee g_j)$ is defined by $h_{ij}(x) = f_i(x)$ if $x \in X_1$ and $h_{ij}(x) = g_j(x)$ if $x \in X_2$.
(3) the probability $p(h_{ij}) = c_i \vee d_j$, that is $p(h_{ij}) = c_i$ if $h_{ij} = f_i$ or $p(h_{ij}) = d_j$ if $h_{ij} = g_j$.

If T_1 and T_2 are the transition matrices of $\mathcal{X}_1$ and $\mathcal{X}_2$ respectively, Then $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ is the transition matrix of $\mathcal{X}_1 \oplus \mathcal{X}_2$.

Example 3.1.

An example of sum is the PRN obtained by summing the same PRN twice, $\mathcal{X} \oplus \mathcal{X}$. To make the disjoint union, we subindicate X with 0 for the first X and with 1 for the second X. That is, the new set is

$$X_0 \dot\cup X_1 = \{(0,0,0), (0,1,0), (0,1,0), (1,1,0)\}$$

$$\cup \{(0,0,1), (0,1,1), (0,1,1), (1,1,1)\}.$$

The digraph is:

$$
\begin{array}{ccccc}
\curvearrowright^{.6} & & \curvearrowright^{.4} & & \curvearrowright^{1} \\
(1,1,0) & \to^{.4} & (1,0,0) & \to^{.6} & (0,0,0) \\
& & & & \curvearrowright^{1} \\
& & & & (0,1,0)
\end{array}
$$

$$
\begin{array}{ccccc}
\curvearrowright^{.6} & & \curvearrowright^{.4} & & \curvearrowright^{1} \\
(1,1,1) & \to^{.4} & (1,0,1) & \to^{.6} & (0,0,1) \\
& & & & \curvearrowright^{1} \\
& & & & (0,1,1)
\end{array}
$$

This is a way to construct a PRN over $\{0,1\}^n$ using either one or two PRN over $\{0,1\}^{n-1}$, since $2^{n-1} + 2^{n-1} = 2^n$.

Superposition

It is clear that a PRN is the **superposition** of several Finite dynamical Systems (FDS)[10] over the same set X with probabilities assigned to each FDS. Since each functions defined over a finite field can be wrote as a polynomial function, we will use this notation for functions over a finite field, [5]. If $X = \{0,1\} = \mathbb{Z}_2$, the finite field of two elements, all the FDSs over X have one of the following state space, where $f_1(x) = x$; $f_2(x) = 1$; $f_3(x) = 0$; $f_4(x) = x + 1$, $\forall x \in X$:

$$
\begin{array}{cccc}
L_1 & L_2 & L_3 & L_4 \\
0 \circlearrowleft & 0 & 0 \circlearrowleft & 0 \\
& \downarrow & \uparrow & \uparrow\downarrow \\
1 \circlearrowleft & 1 \circlearrowleft & 1 & 1
\end{array}
$$

If p_i denotes de probability assigned to L_i, and T_i denotes its transition matrix, then the set of all PRN is described as follows.

$$\left\{ (X, F, C) | T = \sum_{i=1}^{4} p_i T_i = \begin{pmatrix} p_1 + p_3 & p_2 + p_4 \\ p_3 + p_4 & p_1 + p_2 \end{pmatrix} \sum_{i=1}^{4} p_i = 1 \right\}$$

We denote by $L_1 L_2$ the superposition of L_1 and L_2, and similarly $L_1 L_3$ is the superposition of L_1 and L_3. The following state spaces are the superposition of two FDS with two elements:

$$
\begin{array}{cccccc}
L_1 L_2 & L_1 L_3 & L_1 L_4 & L_2 L_3 & L_2 L_4 & L_3 L_4 \\
0 \circlearrowleft^{p_1} & 0 \circlearrowleft^{1} & 0 \circlearrowleft^{p_1} & 0 \circlearrowleft^{p_3} & 0 & 0 \circlearrowleft^{p_3} \\
\downarrow^{p_2}\ , & \uparrow^{p_3}\ , & {}^{p_4}_{\uparrow\downarrow}\ , & \downarrow^{p_2}\uparrow^{p_3}\ , & \downarrow^{p_4}\uparrow^{p_2+p_4}\ , & {}^{p_4}_{\downarrow}\uparrow^{p_4+p_3}\ , \\
1 \circlearrowleft^{1} & 1 \circlearrowleft^{p_1} & 1 \circlearrowleft^{p_1} & 1 \circlearrowleft^{p_3} & 1 \circlearrowleft^{p_3} & 1
\end{array}
$$

For example, with transition matrices

$$T_{12} = T_1 + T_2 = \begin{pmatrix} p_1 & p_2 \\ 0 & 1 \end{pmatrix} \quad T_{13} = T_1 + T_3 = \begin{pmatrix} 1 & 0 \\ p_3 & p_1 \end{pmatrix}$$

Product of two PRN

Let $\mathcal{X}_1 = (X_1, F = (f_i)_{i=1}^{n}, C)$ and $\mathcal{X}_2 = (X_2, G = (g_j)_{j=1}^{m}, D)$ be two PRN. The product $\mathcal{X}_1 \times \mathcal{X}_2 = (X_1 \times X_2, F \times G, C \wedge D)$ is a PRN where

(1) $X_1 \times X_2$ is the cartesian product of X_1 and X_2.
(2) the function $h_{ij} = (f_i, g_j)$ is defined by

$$h_{ij}(x_1, x_2) = (f_i(x_1), g_j(x_2))$$

for $x_1 \in X_1$, and $x_2 \in X_2$.
(3) the probability $p(h_{ij})$ is a function of c_i and d_j, for example $p(h_{ij}) = \frac{c_i + d_j}{2}$.

Example 3.2.

The product $L_1 L_2 \times L_1 L_3$ is the PRN with four states $\{(0,0), (0,1), (1,0), (1,1)\}$ and four functions

$$f_{11}(x, y) = (x, y), f_{13}(x, y) = (x, 0),$$

$$f_{21}(x, y) = (1, y), \ f_{23}(x, y) = (1, 0).$$

The state space is the following:

$$
\begin{array}{c}
L_1 L_2 \times L_1 L_3 \\
{}^{p_{11}+p_{13}} \circlearrowleft (0,0) \xleftarrow{p_{13}} (0,1) \circlearrowleft^{p_{11}} \\
{}^{p_{23}+p_{21}} \downarrow \quad \swarrow_{p_{23}} \downarrow^{p_{21}} \\
\circlearrowleft^{1} (1,0) \xleftarrow{} (1,1) \circlearrowleft^{p_{11}+p_{21}} \\
{}_{p_{13}+p_{23}}
\end{array}
$$

The transition matrix is the following

$$T = \begin{pmatrix} p_{11} + p_{13} & 0 & p_{23} + p_{21} & 0 \\ p_{13} & p_{11} & p_{23} & p_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & p_{13} + p_{23} & p_{11} + p_{21} \end{pmatrix}$$

3.1. **Linear Probabilistic Regulatory Networks.** A linear PRN is a **superposition** of linear FDS. A linear FDS is a pair (X, f) where f is a linear function, and X is a vector space over a finite field. So, a linear PRN is a triple $(X, (f_i)_{i=1}^m, C)$, where X is a finite vector space, the functions $f_i : X \to X$ are linear functions, and $C = \{c_i = p(f_i)\}$. The set X has cardinality a power of a prime number and each linear function is determined by its characteristic polynomial and the companion matrix.

If $X = \mathbb{Z}_3 = \{0, 1, 2\}$ is the field of integer modulo 3, then the linear functions are: $f_1(x) = x$, $f_2(x) = 2x$, and $f_3(x) = 0$ for all $x \in \mathbb{Z}_3$. So, the linear PRN are the following:

$$
\begin{array}{cccc}
\{f_1, f_2\} & \{f_1, f_3\} & \{f_2, f_3\} & \{f_1, f_2, f_3\} \\
\circlearrowleft^1 & \circlearrowleft^1 & \circlearrowleft^1 & \circlearrowleft^1 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
& \nearrow^{p_3}\nwarrow & \nearrow^{p_3}\nwarrow & \nearrow^{p_3}\nwarrow \\
\circlearrowleft^{p_1} \mathbf{1} \underset{p_2}{\overset{}{\rightleftarrows}} \mathbf{2} \circlearrowleft^{p_1} & \mathbf{1} \circlearrowleft_{p_1} \quad \circlearrowleft_{p_1} \mathbf{2} & \mathbf{1} \rightleftarrows_{p_2} \mathbf{2} & \circlearrowleft_{p_1} \mathbf{1} \rightleftarrows_{p_2} \mathbf{2} \circlearrowleft_{p_1}
\end{array}
$$

If $X = \mathbb{Z}_2 \times \mathbb{Z}_2$ is the vector space with 4 elements over the field $\mathbb{Z}_2$, then there are 4 linear FDS not isomorphics. In fact, using matrix, the possible characteristics polynomials $p_f(\lambda)$ are: λ^2, $\lambda^2 + \lambda$ $\lambda^2 + 1$, $\lambda^2 + \lambda + 1$. The companion matrices of these linear functions are:

$$
A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad A_4 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}
$$

Then the FDS associated to this matrices are:

$$
\begin{array}{cc}
A_1 & A_2 \\
\circlearrowleft (0,0) \;\leftarrow\; (1,0) & \circlearrowleft (0,0) \;\leftarrow\; (1,0) \\
\uparrow \qquad \nwarrow & \\
(0,1) \qquad (1,1) & \circlearrowleft (0,1) \;\leftarrow\; (1,1)
\end{array}
$$

$$
\begin{array}{cc}
A_3 & A_4 \\
\circlearrowleft (0,0) \quad \circlearrowleft (1,0) & \circlearrowleft (0,0) \qquad (1,0) \\
& \swarrow \quad \uparrow \\
\circlearrowleft (0,1) \quad \circlearrowleft (1,1) & (0,1) \;\rightarrow\; (1,1)
\end{array}
$$

The linear PRN with two functions are the following:

$$
\begin{array}{ccc}
A_1, A_2 & A_1, A_3 & A_1, A_4 \\
\overset{1}{\circlearrowleft}(0,0)\overset{1}{\leftarrow}(1,0) & \overset{1}{\circlearrowleft}(0,0)\underset{}{\overset{p_3}{\rightleftarrows}}(1,0)\overset{p_3}{\circlearrowleft} & \circlearrowleft^1 (0,0)\overset{p_1}{\leftarrow}(1,0) \\
{}^{p_1}\uparrow\nwarrow{}^{p_1} & {}^{p_1}\uparrow\nwarrow{}^{p_1} & {}^{p_1}\uparrow \;\searrow\; \uparrow^{p_4} \\
\overset{p_2}{\circlearrowleft}(0,1)\overset{p_2}{\leftarrow}(1,1) & {}^{p_3}\circlearrowleft\;(0,1)\;(1,1)\;\circlearrowleft^{p_3} & (0,1)\overset{p_4}{\leftarrow}(1,1)
\end{array}
$$

$$
\begin{array}{ccc}
A_2, A_3 & A_2, A_4 & A_3, A_4 \\
\circlearrowleft^1 (0,0) \leftarrow^{p_2} (1,0) \circlearrowleft^{p_3} & \circlearrowleft^1 (0,0) \leftarrow^{p_2} (1,0) & \circlearrowleft^1 (0,0) \;(1,0) \circlearrowleft^{p_3} \\
& {}^{p_4}\swarrow\uparrow^{p_4} & {}^{p_4}\swarrow\uparrow^{p_4} \\
\circlearrowleft^1 (0,1) \leftarrow^{p_2} (1,1) \circlearrowleft^{p_3} & \circlearrowleft^{p_2} (0,1) \leftarrow^{p_4} (1,1) & {}^{p_3}\circlearrowleft (0,1) \rightarrow^{p_4} (1,1) \circlearrowleft^{p_3}
\end{array}
$$

Example 3.3. *Looking the state spaces of the A_i, A_j for $i \neq j$ PRN, we can conclude that two of them are not isomorphic. Working with the transition matrices, we can calculate when there exists an homomorphism between two of them. For example: the set $Hom(A_1 A_2; A_1 A_3) = \emptyset$*

Example 3.4.

If X_1 and X_2 are two PRNs then the product $X_1 \times X_2$ can be projected over each component. In the example of product of two PRN, we use the PRN $L_1 L_2$ and $L_1 L_3$ to construct $L_1 L_2 \times L_1 L_3$. The functions $\phi_1(x, y) = x$ and $\phi_2(x, y) = y$, the usual projections, are ϵ-homomorphism. In fact

(1)$\phi_i : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2$, and $\phi_1 \circ f_{11} = f_1 \circ \phi_1$,and $\phi_1 \circ f_{21} = f_2 \circ \phi_1$.

(2) the second condition is satisfies too, since several arrows compact to one arrow in $\mathbb{Z}_2$, for example $\phi_1(0, 1) = \phi_1(0, 0) = 0$, then the arrows compact in the following way

$$\{^{p_{11}+p_{13}}\circlearrowleft (0,0) \leftarrow^{p_{13}} (0,1) \circlearrowright^{p_{11}}\} \quad \twoheadrightarrow \quad 0 \circlearrowright^{p_1}$$

Similarly

$$\{\circlearrowleft^1 (1,0) \leftarrow^{p_{13}+p_{23}} (1,1) \circlearrowright^{p_{11}+p_{21}}\} \quad \twoheadrightarrow \quad 1 \circlearrowleft 1$$

$$\left\{ \begin{array}{cc} (0,0) & (0,1) \\ \downarrow{p_{23}+p_{21}} \quad \swarrow_{p_{23}}\downarrow{p_{21}} \\ (1,0) & (1,1) \end{array} \right\} \quad \twoheadrightarrow \quad \begin{array}{c} 0 \\ \downarrow{p_2} \\ 1 \end{array}$$

(3) for the ϵ-condition, the probabilities need to satisfy:

$$|p_{11} - p_1|, \; |p_{13} - p_1|, \; |(p_{11} + p_{13}) - p_1| < \epsilon$$

$$|(p_{13} + p_{23}) - 1|, \; |(p_{11} + p_{21}) - 1| < \epsilon$$

$$|p_{21} - p_2|, \; |p_{23} - p_2|, \; |(p_{21} + p_{23}) - p_2| < \epsilon$$

since $p_1 + p_2 = 1$, and $p_1 + p_3 = 1$, for $p_{ij} = p_i p_j$ for example, we have $|(p_{11} + p_{13}) - p_1| = |(p_{21} + p_{23}) - p_2| = 0$, $|p_{11} - p_1| = |p_{23} - p_2| = |p_1 p_2|$, $|p_{13} - p_1| = |p_1^2|$, $|p_{21} - p_2| = |p_2^2|$, $|(p_{13} + p_{23}) - 1| = |p_1|$, and $|(p_{11} + p_{21}) - 1| = |p_2| = |p_3|$. So, $\max(p_1, p_2, p_3) = \epsilon$.

We have the same conditions for ϕ_2. It is clear that always the projections are homomorphisms, but the applications of the ϵ-condition depend on the particular case that we are working. It is clear that there are several ways to define inclusions $\iota : L_1 L_3 \to L_1 L_3 \times L_1 L_2$.

The following digraph shows the projections in general.

$$\begin{array}{ccc} & X_1 \times X_2 & \\ \swarrow{\pi_1} & \pi_2 & \searrow \\ X_1 & & X_2 \end{array}$$

The functions $\pi_i : X_1 \times X_2 \to X_i$, are defined are follows: $\pi_i(x_1, x_2) = x_i$, $i = 1, 2$. It is easy to check that the others two conditions are satisfied as in the above example.

4. Invariant Subnetworks and Projections

A subnetwork $Y \subseteq X$ of $\mathcal{X} = (X, F, C)$ is an **invariant subnetwork or a sub-PRN** of $\mathcal{X}$ if $f_i(u) \in Y$ for all $u \in Y$, and $f_i \in F$. Sub-PRNs are sections of a PRN, where there aren't arrows going out. The complete network X, and any cyclic state with probability 1, are sub-PRNs. An invariant subnetwork is irreducible if doesn't have a proper invariant subnetwork. *An endomorphism is a projection if $\pi^2 = \pi$.*

Theorem 4.1. *If there exists a projection from $\mathcal{X}$ to a subnetwork $\mathcal{Y}$ then $\mathcal{Y}$ is an invariant subnetwork of $\mathcal{X}$.*

Proof. Suppose that there exists a projection $\pi : X \to Y$. If $y \in Y$, by definition of projection $\pi(y) = y$, and $f_i(\pi(y)) = \pi(g_j(y))$. Therefore all arrows in the subnetwork Y are going inside Y, and the network is invariant. $\qquad\square$

Example 4.2.

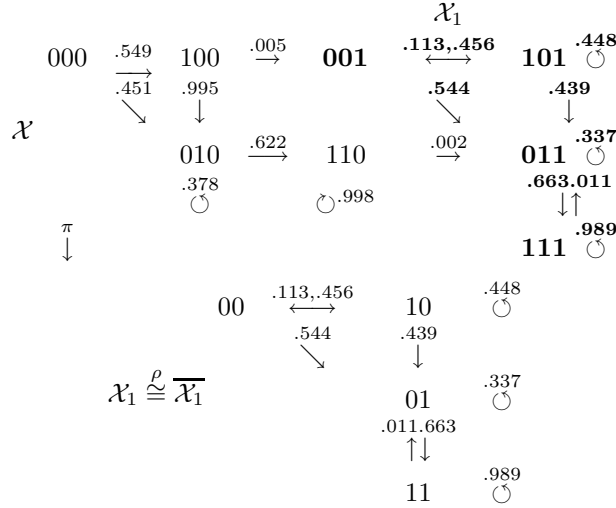The PRN $\overline{\mathcal{X}}$ has two invariant subnetworks with projections
$\pi_1(x,y) = (x,0)$ and $\pi_2(x,y) = (1,y)$.

$$\overset{.67}{\circlearrowleft}(0,0) \xleftarrow{.21}(0,1)\overset{.46}{\circlearrowright}$$

$$.33 \downarrow \quad \overset{.11}{\nearrow}\downarrow .22$$

$$\circlearrowleft^1 (1,0) \xleftarrow{.32}(1,1)\overset{.68}{\circlearrowright}$$

$$\overline{S}_1 \ \overset{.67}{\underset{.33}{\circlearrowleft}} \begin{matrix}0\\\downarrow\\\end{matrix} \quad \cong \quad \overset{.67}{\underset{.33}{\circlearrowleft}} \begin{matrix}(0,0)\\\downarrow\\\end{matrix} \ S_1 \xleftarrow{\pi_1} \quad \begin{matrix}\overline{\mathcal{X}}\\ \pi_2 \\ \downarrow \end{matrix}$$

$$\circlearrowleft^1 1 \qquad \circlearrowleft^1 (1,0)$$

$$S_2 \ \circlearrowleft^1 (1,0) \xleftarrow{.32}(1,1)\overset{.68}{\circlearrowright}$$

$$\cong$$

$$\overline{S}_2 \ \circlearrowleft^1 0 \xleftarrow{.32} 1 \overset{.68}{\circlearrowright}$$
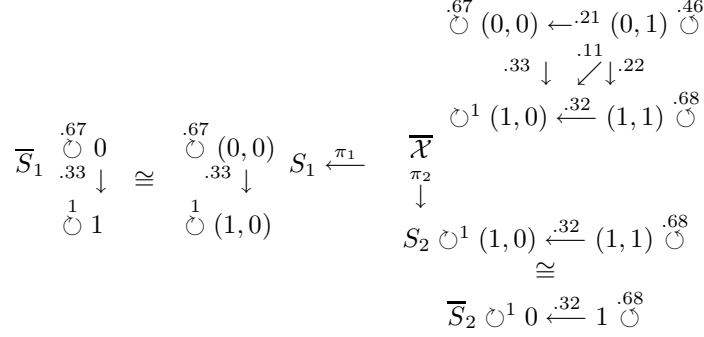
Checking the probabilities for π_1 and π_2, we have $\epsilon_1 = .68$; and $\epsilon_2 = .67$. We can observe that $\overline{\mathcal{X}} \cong \overline{S}_1 \times \overline{S}_2$.

Example 4.3.

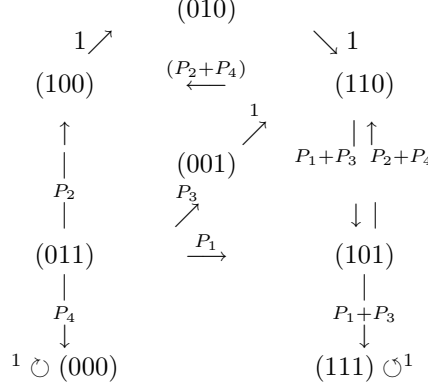The subnetwork $\mathcal{X}_1 = (\{(x,y,1)\}, F, C)$ is an invariant subnetwork of $\mathcal{X} = (\{0,1\}^3, F, C)$.

$$\mathcal{X}_1$$

$$000 \xrightarrow[.451]{.549} 100 \xrightarrow{.005} \mathbf{001} \xleftrightarrow{.113,.456} \mathbf{101}\overset{.448}{\circlearrowright}$$

$$\mathcal{X} \qquad \searrow \quad \downarrow .995 \qquad \mathbf{.544} \qquad .439 \downarrow$$

$$010 \xrightarrow{.622} 110 \xrightarrow{.002} \mathbf{011}\overset{.337}{\circlearrowright}$$

$$.378\circlearrowleft \qquad \circlearrowright .998 \qquad \mathbf{.663.011} \\ \downarrow\uparrow$$

$$\begin{matrix}\pi\\\downarrow\end{matrix} \qquad \qquad \mathbf{111}\overset{.989}{\circlearrowright}$$

$$00 \xleftrightarrow{.113,.456} 10 \overset{.448}{\circlearrowright}$$

$$.544 \qquad .439 \downarrow$$

$$\mathcal{X}_1 \overset{\rho}{\cong} \overline{\mathcal{X}}_1 \qquad \searrow \qquad 01 \overset{.337}{\circlearrowright}$$

$$.011.663 \\ \uparrow\downarrow$$

$$11 \overset{.989}{\circlearrowright}$$

Ordering the elements in the following way $\{(0,0,0), (0,1,0), (1,0,0), (1,1,0), (0,0,1), (0,1,1), (1,0,1), (1,1,1)\}$, the matrix

$$T_{X_1} = \begin{bmatrix} 0 & .544 & .456 & 0 \\ 0 & .337 & 0 & .663 \\ .113 & .448 & .439 & 0 \\ 0 & .011 & 0 & .989 \end{bmatrix} \text{ is an invariant part of the transition matrix}$$

$$T_{\mathcal{X}} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{X_1} \end{bmatrix}.$$

Using the projection $\pi : \mathcal{X} \to \mathcal{X}_1$, $\pi(x,y,z) = (x,y,1)$; and the isomorphism $\rho(x,y,1) = (x,y)$, the network $\mathcal{X}$ is projected over the network $\overline{\mathcal{X}_1}$. Checking the arrows the projection $\overline{\pi}$ is a .5-homomorphism.

Example 4.4. *The following PBN appears in* [11], *and has three sub-PBNs.*

$$
\begin{array}{ccc}
& (010) & \\
1 \nearrow & & \searrow 1 \\
(100) & \xleftarrow{(P_2+P_4)} & (110) \\
\uparrow & & | \uparrow \\
| & \overset{1}{\nearrow} & \\
P_2 & (001) \quad P_3 & P_1+P_3 \;\; P_2+P_4 \\
| & \nearrow & \downarrow | \\
(011) & \xrightarrow{P_1} & (101) \\
| & & | \\
P_4 & & P_1+P_3 \\
\downarrow & & \downarrow \\
{}^1 \circlearrowleft (000) & & (111) \circlearrowright^1
\end{array}
$$

$X_1 = \{(000)\}$, $X_2 = \{(111)\}$, and $X_3 = \{(100),(010),(110),(101),(111)\}$ are sub networks. With adequate order the transition matrices T_X and T_{X_3} are:

$$
T_{X_3} = \begin{bmatrix}
0 & 0 & .5 & .5 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
.5 & 0 & 0 & 0 & .5 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix} \text{ and } \pi_{X_3} = (0,0,0,0,1)
$$

meanwhile $T_X = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{X_3} \end{bmatrix}$, and $\pi_X = (.25,0,0,0,0,0,0,.75)$.

5. The category of Probabilistic Regulatory Networks, and mathematical background

Theorem 5.1. *If* $\phi_1 : \mathcal{X}_1 \to \mathcal{X}_2$ *is an* ϵ_1-*homomorphism, and* $\phi_2 : \mathcal{X}_2 \to \mathcal{X}_3$ *is another* ϵ_2-*homomorphism. Then* $\phi = \phi_2 \circ \phi_1 : \mathcal{X}_1 \to \mathcal{X}_3$ *is an* ϵ-*homomorphism. Therefore the Probabilistic Regulatory Networks with the homomorphisms of PRN form the category* **PRN**.

Proof. The Probabilistic Regulatory Networks with the PRN homomorphisms is a category if: the composition is an homomorphism, and satisfy the associativity law; and there exists an identity homomorphism for each PRN.

(1) Let $\phi_1 : \mathcal{X}_1 \to \mathcal{X}_2$ be an ϵ_1-homomorphism, and let $\phi_2 : \mathcal{X}_2 \to \mathcal{X}_3$ be an ϵ_2-homomorphism. If q_t, g_k and f_j are functions in each PRN, and such that $\phi_1 \circ f_j = g_k \circ \phi_1$ and $\phi_2 \circ g_k = q_t \circ \phi_2$, then we will prove that: $\phi \circ f_j = q_t \circ \phi$. In fact,

$$
(\phi_2 \circ \phi_1) \circ f_j = \phi_2 \circ (\phi_1 \circ f_j) =
$$
$$
\phi_2 \circ (g_k \circ \phi_1) = (\phi_2 \circ g_k) \circ \phi_1 =
$$
$$
(q_t \circ \phi_2) \circ \phi_1 = q_t \circ (\phi_2 \circ \phi_1).
$$

(2) We want to prove that

$$
\chi(t_k(\phi(u),\phi(v))) \geq \chi(c_i(u,v)).
$$

Suppose that $\chi(c_i(u,v)) = 1$. Then, since ϕ_1 is an homomorphism of PRN, we have that

$$\chi(d_j(\phi_1(u), \phi_1(v))) \geq \chi(c_i(u,v))$$

which is 1. Since ϕ_2 is an homomorphism of PRN, we obtain that

$$\chi(t_k(\phi(u), \phi(v))) = \chi(t_k(\phi_2(\phi_1(u)), \phi_2(\phi_1(v))))$$

$$\geq \chi(c_j(\phi_1(u), (\phi_1(v)) = 1.$$

Therefore we obtain that

$$\chi(t_k(\phi_2(\phi_1(u)), \phi_2(\phi_1(v)))) = 1.$$

Then the composition of two PRN-homomorphisms is an homomorphism.

(3) To verify the third condition for ϵ-homomorphism, we do the following. If $p(\phi(u_1), \phi(u_2)) > 1$, with $u_1,\ u_2 \in X_1$, then we need to prove that there exists an ϵ such that

$$|p(u_1, u_2) - p(\phi(u_1), \phi(u_2))| < \epsilon.$$

In fact:

$$|p(u_1, u_2) - p(\phi(u_1), \phi(u_2))| = |p(u_1, u_2) - p(\phi_1(u_1), \phi_1(u_2)) +$$

$$p(\phi_1(u_1), \phi_1(u_2)) - p(\phi_2(\phi_1(u_1)), \phi_2(\phi_1(u_2)))|$$

$$< |p(u_1, u_2) - p(\phi_1(u_1), \phi_1(u_2))| +$$

$$|p(\phi_1(u_1), \phi_1(u_2)) - p(\phi_2(\phi_1(u_1)), \phi_2(\phi_1(u_2)))| \leq \epsilon_1 + \epsilon_2$$

$$|p(u_1, u_2) - p(\phi(u_1), \phi(u_2))| < \epsilon_1 + \epsilon_2$$

because ϕ_1 and ϕ_2 are ϵ-homomorphisms.

The associativity and identity laws are easily checked, therefore our claim holds, and **PRN** is a category. $\qquad \square$

It is clear that, the PRN with the homomorphism between them form a category that we will denote $\mathcal{PRN}$. The category **PRN** is a subcategory of $\mathcal{PRN}$, since an homomorphism is not always an homomorphism for some $\epsilon \in \mathbb{R}$ enough small. But, if we don't include the condition for ϵ to be enough small, the two categories are the same, because always an homomorphism is an ϵ-homomorphism for some $\epsilon \in \mathbb{R}$.

Theorem 5.2. *Let $\mathcal{X}_1 \times \mathcal{X}_2 = (X_1 \times X_2, H, E)$ be a product of PRN $\mathcal{X}_1 = (X_1, F, C)$ and $\mathcal{X}_2 = (X_2, G, D)$. If $\delta_i : X \to X_i$ are two PRN-homomorphisms, then there exists an homomorphism $\delta : X \to X_1 \times X_2$, such that $\phi_i \circ \delta = \delta_i$ for $i = 1, 2$. That is, the following diagram commutes*

$$
\begin{array}{ccc}
 & X_1 \times X_2 & \\
{\scriptstyle\phi_1}\swarrow \ \ {\scriptstyle\delta}\uparrow \ \ {\scriptstyle\phi_2}\searrow & & \\
X_1 \ \xleftarrow{\ \delta_1\ } \ X \ \xrightarrow{\ \delta_2\ } \ X_2 &
\end{array}
$$

This homomorphism is unique.

Proof. The function $\delta : X \to X_1 \times X_2$ is defined as follows $\delta(x) = (\delta_1(x), \delta_2(x))$, $x \in X$. δ is an homomorphism, in fact:

(1) Let $\mathcal{X} = (X, L, P)$ be a PRN. Since δ_1 and δ_2 are homomorphism, for all function $l_t \in L$ there exist two functions $f_i \in F$ and $g_j \in G$, such that $\delta_1 \circ l_t = f_i \circ \delta_1$, and $\delta_2 \circ l_t = g_j \circ \delta_2$. Then for the function l_t there exists the function (f_i, g_j) that satisfies $\delta \circ l_t = (f_i, g_j) \circ \delta$.

$$(\delta \circ l_t)(x) = \delta(l_t(x)) = (\delta_1(l_t(x)), \delta_2(l_t(x))) =$$

$$(f_i(\delta_1(x)), g_j(\delta_2(x))) = ((f_i, g_j) \circ \delta)(x)$$

(2) In order to prove $\chi(e_{ij}(\delta(x), \delta(x'))) \geq \chi(p_{l_t}(x, x'))$, suppose $\chi(p_{l_t}(x, x')) = 1$. Then $l_t(x) = x'$, and $\delta(x') = \delta(l_t(x)) = (f_i, g_j)(\delta(x))$ by part (1). Therefore $\chi(e_{ij}(\delta(x), \delta(x'))) = 1$, and our claim holds.

It is easy to check that $\phi_i \circ \delta = \delta_i$, in fact

$$\phi_1(\delta(x)) = \phi_1(\delta_1(x), \delta_2(x)) = \delta_1(x),$$

for all $x \in X$. $\square$

If δ_i, $i = 1, 2$, are ϵ_i-homomorphism then

$$max|p(x, x') - p(\phi_1(\delta(x)), \phi_1(\delta(x')))| \leq \epsilon_1.$$

But

$$|p(x, x') - p(\delta(x), \delta(x')) + p(\delta(x), \delta(x')) - p(\phi_1(\delta(x)), \phi_1(\delta(x')))| \leq$$

$$|p(x, x') - p(\delta(x), \delta(x'))|+$$

$$|p(\delta(x), \delta(x')) - p(\phi_1(\delta(x)), \phi_1(\delta(x')))| \leq \epsilon_1.$$

Therefore

$$|p(x, x') - p(\delta(x), \delta(x'))| \leq \epsilon_1 - |p(\delta(x), \delta(x')) - p(\phi_1(\delta(x)), \phi_1(\delta(x')))|$$

$$|p(x, x') - p(\delta(x), \delta(x'))| \leq \epsilon_1 - \overline{\epsilon}_1.$$

Therefore δ is an ϵ-homomorphism. So, the theorem holds for ϵ-homomorphism.

It is an immediate consequence the following result, also is true for ϵ-homomorphisms.

Theorem 5.3. *Let* $\mathcal{X}_1 \oplus \mathcal{X}_2 = (X_1 \times X_2, H, E)$ *be a product of PRN* $\mathcal{X}_1 = (X_1, F, C)$ *and* $\mathcal{X}_2 = (X_2, G, D)$. *If* $\gamma_i : X_i \to X$ *are two PRN-homomorphisms, then there exists an homomorphism* $\gamma : X_1 \oplus X_2 \to X$, *such that* $\gamma \circ \iota_i = \gamma_i$ *for* $i = 1, 2$. *That is, the following diagram commutes*

$$X_1 \oplus X_2$$
$$\begin{array}{ccc} {}^{\iota_1}\nearrow & {}^{\gamma}\downarrow & {}^{\iota_2}\nwarrow \\ X_1 \xrightarrow{\gamma_1} & X & \xleftarrow{\gamma_2} X_2 \end{array}$$

This homomorphism is unique.

Theorem 5.4. *All reducible PRN is either a product of its non trivial sub-PRN or a subnetwork of this product.*

Proof. It is trivial by definition of Product and sub-PRN. $\square$

6. Conclusions

The intersection, and the union of two sub-PRN is a sub-PRN, therefore the class of sub-PRN of a particular PRN is a lattice. Reduction mappings described in [11] and defined for PBN using the influence of a gene, for example x_n, on the predictor function $f_j^{(i)}$ to determine the selected predictor, can be extended to PRN. In order to extend this procedure to more than boolean functions, we use the polynomial description of genetic functions given in [5], the partial derivative is the usual in calculus and all the concepts in [11] can be using for PRN. Similarly our definition of projection, the reduction mappings are ϵ-homomorphisms, and we can use for genes with more than two quantization, since this extension is not a trivial work we develop the theory and methods in [1].

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Department of Mathematic-Physics, University of Puerto Rico, Cayey, PR 00736

E-mail address: mavino@cayey.upr.edu